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Onset of convection in a porous medium with nonuniform time-dependent volumetric heating

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Linear stability analysis is applied to study the onset of convection, induced by volumetric source heating or cooling, in a horizontal layer occupied by a saturated porous medium. The source strength is assumed to vary exponentially with depth and vary with time; the time dependence is taken in turn to be sinusoidal or square-wave periodic, and a transient situation is also investigated. A vertical applied temperature gradient may also be present. Various combinations of thermal boundary conditions are considered. Analytical expressions that give upper bounds on an appropriate critical Rayleigh number are derived for each case.

Keywords: porous medium; natural convection; volumetric heating; time-dependent heating

Introduction

This paper is directed toward two related goals. One is to model analytically the problem of convection, in a horizontal layer of a saturated porous medium, induced by time-periodic heating by radiation incident on the upper surface. The second goal is to extend the study of the classical Horton–Rogers–Lapwood (HRL) problem (the onset of convection with uniform heating from below) toward the case of general time-dependent heating.

A number of practical applications have motivated this investigation. Solar energy collectors [and also solar ponds (Hadim and Burmeister, 1992)] are built using porous materials. Grains are frequently packed inside rectangular glass containers for drying or cooking by solar radiation. The thermal insulation of flat roofs subject to solar incidence is of interest. The accumulation of nutrients within the top levels of sand in shallow water subject to solar heating is of biological importance.

There is much literature on the HRL problem [see, for example, Chapter 6 of Nield and Bejan (1992)], but to the author's knowledge there are few published papers concerned with time-dependent heating. Some of these are concerned with situations where the imposed surface temperature varies monotonically with time in an unbounded fashion. Now, amplification of disturbances inevitably occurs at some stage, and the interest is in determining an onset time by which the growth factor has reached a specified ratio, say, 1,000. Caltagirone (1980) investigated the case where the lower surface is subject to a sudden rise of temperature; he used linear theory, energy-based theory, and a two-dimensional (2-D) numerical model. Kaviany (1984a) made a theoretical and experimental investigation of a layer with a lower surface temperature increasing linearly with time. His second paper (1984b) involved both time-dependent cooling of the upper surface and uniform internal heating.

Four papers have dealt with time-periodic heating. Rudraiah and Malashetty (1990) treated the case of a modulated applied temperature gradient. Chhuon and Caltagirone (1979) examined the case where the temperature imposed on the boundary is time-periodic, with a nonzero mean value. They performed experiments and compared their observations with those obtained when Floquet theory was used to examine the stability of solutions of the ordinary differential equation system governing the perturbed variables, and also with calculations, based on linear theory applied to a "frozen" profile, obtained earlier by Caltagirone (1976). McKay (1992) was concerned with patterned ground formation as a result of solar radiation ground heating. Accordingly, he applied Floquet theory to the equations pertaining to the case where at the upper surface a time-periodic temperature gradient is specified. [He also allowed for a quadratic relationship between density and temperature (appropriate for icy water) and for a permeability varying with depth.]

In this paper, it is assumed that, as a result of radiation incident from above or from a stratified distribution of heat producing material, there is volumetric heating whose source magnitude decreases exponentially with depth and oscillates periodically with time. In addition, a constant temperature difference is imposed across the layer. The stability of the consequent conduction solution is examined using linear stability analysis based on the assumption of a "frozen" basic temperature profile; i.e. one that varies sufficiently slowly with time (compared with the growth of a disturbance) so it can be taken as constant for the perturbation analysis. This approach, rather than that using Floquet theory, was chosen because the present interest is in whether motion results at any stage of the cycle, rather than whether there is net growth in the magnitude of a disturbance during a complete cycle. Whereas the Floquet-type approach is the method of choice for similar

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problems in clear (of solid material) fluids, the frozen-profile approach seems to be the better one for flow in a dense porous medium, a consequence of the fact that the magnitude of the time-derivative inertial term in the momentum equation is usually small [see, for example, Section 1.5.1 of Nield and Bejan (1992)]. This statement is borne out by the fact that the critical Rayleigh number values observed by Chhuon and Caltagirone (1979) were in better accord with linear theory based on a frozen profile than with Floquet theory. The assumption of a frozen profile leads to a simplified analysis in this paper. [The result of the assumption is that the problem treated is somewhat similar to that treated by Rudraiah et al. (1980, 1982), who studied the effect of time-independent uniform volumetric heating on the onset of convection using a Brinkman model]. A further simplification results from the use of a relatively crude Galerkin approximation in this exploratory investigation, which includes a comparison of the results of various forms of periodic heating (sinusoidal and square-wave) and for various combinations of thermal boundary conditions.

Basic equations

Relative to a Cartesian frame with the z^*-axis vertically upward, a saturated porous medium is assumed to lie between the planes $z^* = 0$ and $z^* = H$. The Oberbeck–Boussinesq approximation, Darcy's law, and local thermal equilibrium are assumed. Accordingly [compare Equations 6.3-6 of Nield and Bejan (1992)], the governing equations for convection in a porous medium are taken to be as follows:

$$\nabla \cdot \mathbf{v}^* = 0 \tag{1}$$

$$c_a \rho_0 \, \partial \mathbf{v}^*/\partial t^* = -\nabla P^* - (\mu/K)\mathbf{v}^* + \rho_f g. \tag{2}$$

$$(\rho c)_m \, \partial T^*/\partial t^* + (\rho c_p)_f \mathbf{v}^* \cdot \nabla T^* = k_m \nabla^2 T^* + q''' e^{\beta(z^*/H - 1)} \{1 + \varepsilon e^{i\omega(t^* - t'')}\} \tag{3}$$

$$\rho_f = \rho_0[1 - \gamma_T(T^* - T_0)] \tag{4}$$

The distinctive new feature is the appearance, in the energy Equation 3, of a volumetric source term, time-periodic and with an amplitude that decays exponentially with depth below the upper surface. The parameters appearing are the dimensionless decay coefficient β, the amplitude ε, the angular frequency ω, and the epoch time t''. (In the context of heating caused by the absorption of radiation, β/H is the attenuation coefficient.) The epoch has been introduced because later, in the linear stability analysis, the basic thermal profile is frozen at its value at $t = 0$, and this is a convenient way of allowing for disturbances of arbitrary phase.

The reader should distinguish between the acceleration coefficient c_a and the dimensionless Forchheimer coefficient c_F [see, for example, Nield and Bejan (1992, p. 9)]. The local acceleration term has been included because it could become significant at large values of ω. Quadratic inertial terms have been excluded because only the onset of convection is treated in the present paper. (J. L. Lage is currently conducting a numerical study in which full nonlinear equations are used.) Because the present investigation is a pioneering one, the Brinkman term has been omitted for the sake of simplicity.

The thermal boundary conditions are taken to be as follows:

$$T^* = T_0 + \Delta T \quad \text{at} \quad z^* = 0; \qquad T^* = T_0 \quad \text{at} \quad z^* = H \tag{5}$$

The boundaries are assumed to be impermeable, so

$$w^* = 0 \quad \text{at} \quad z^* = 0 \quad \text{and at} \quad z^* = H \tag{6}$$

Dimensionless variables are now introduced, defined by the following:

$$(x, y, z) = (x^*, y^*, z^*)/H, \quad \mathbf{v} = \mathbf{v}^* H/\alpha_m, \quad t = t^* \alpha_m/\sigma H^2, \quad T = T^*/T_r, \quad P = P^* K/\mu \alpha_m \tag{7}$$

If $\Delta T \neq 0$, then ΔT is the natural choice for the reference temperature T_r. If $\Delta T = 0$, then T_0 is the appropriate choice.

Notation

c	specific heat
c_a	inertial coefficient
D	differential operator, d/dz
g	gravitational acceleration
H	layer height
i, j, k	unit Cartesian vectors
k	thermal conductivity
K	permeability
l, m	dimensionless wavenumber in the x- and y-directions
P	pressure (excess over hydrostatic)
q'''	volumetric heat source amplitude (Equation 3)
Q	$q''' H^2/k_m T_r$
R	$g\gamma_T K H T_r/\nu\alpha_m$
Ra	external Rayleigh number, $g\gamma_T K H \, \Delta T/\nu\alpha_m$
Ra_I	internal Rayleigh number, $g\gamma_T K H^3 q'''/2\nu\alpha_m k_m$
t	time
t''	epoch (Equation 3)
$\mathbf{v}$	velocity vector $(=(u, v, w))$
x, y, z	Cartesian coordinates

Greek

α	dimensionless overall horizontal wavenumber
α_m	thermal diffusivity, $k_m/(\rho c_p)_f$,
β	dimensionless decay constant (see Equation 3)
γ_a	dimensionless inertial coefficient
γ_T	coefficient of volume expansion
ε	amplitude factor (see Equation 3)
θ	dimensionless temperature perturbation
κ	thermal diffusivity
λ	$(i\Omega)^{1/2}$
μ	dynamic viscosity
ν	kinematic viscosity
ρ	density
σ	heat capacity ratio, $(\rho c)_m/(\rho c_p)_f$
τ	dimensionless epoch
ω	angular frequency
Ω	dimensionless angular frequency, $\sigma H^2 \omega/\alpha_m$

Subscripts

c	critical
0	standard
r	reference

Superscripts

$'$	perturbation
$*$	dimensional variables

The dimensionless numbers that naturally arise are as follows:

$$\lambda = (i\Omega)^{1/2},\quad \Omega = \sigma H^2\omega/\alpha_m,\quad Q = q'''H^2/k_m T_r,$$
$$R = g\gamma_T K H T_r/\nu\alpha_m,\ \gamma_a = c_a\rho\alpha_m K/\sigma\mu H^2,\quad \tau = t''\alpha_m/\sigma H^2 \tag{8}$$

Later, in the stability analysis, the combinations $\mathrm{Ra} = R\Delta T/T_r$ and $\mathrm{Ra}_I = \frac{1}{2}RQ$ arise. Because Ra_I involves heating internal to the porous medium, it is referred to as the internal Rayleigh number. The usual Rayleigh–Darcy number Ra is referred to as the external Rayleigh number. The factor $\frac{1}{2}$ is introduced to accord with the notation in Gasser and Kazimi (1976) and Nield and Bejan (1992, Section 6.11.2).

The basic (conduction) solution $\mathbf{v}_b$, T_b is then given by the following:

$$\begin{aligned}\mathbf{v}_b = 0,\quad T_b &= [T_0 + \Delta T(1 - z)]/T_r \\ &+ (Q/\beta^2)[(1 - e^{-\beta})z + e^{-\beta}(1 - e^{\beta z})] \\ &+ [\varepsilon Q e^{i\Omega(t-\tau)}/(\beta^2 - \lambda^2)\sinh\lambda] \\ &\times \{\sinh\lambda z + e^{-\beta}[\sinh\lambda(1 - z) - e^{\beta z}\sinh\lambda]\}\end{aligned} \tag{9}$$

Stability analysis

Writing $\mathbf{v} = \mathbf{v}_b + \mathbf{v}'$, $T = T_b + T'$, $P = P_b + P'$, and linearizing the resulting equations, we obtain the following:

$$\nabla\cdot\mathbf{v}' = 0 \tag{10}$$

$$\gamma_a\,\partial\mathbf{v}'/\partial t = -\nabla P' - \mathbf{v}' + RT'\mathbf{k} \tag{11}$$

$$\partial T'/\partial t + (\partial T_b/\partial z)w' = \nabla^2 T' \tag{12}$$

Eliminating P', and writing

$$[w', T'] = [w(z, t), \theta(z, t)]\exp\{ilx + imy\} \tag{13}$$

$$\alpha = (l^2 + m^2)^{1/2},\quad D \equiv \partial/\partial z$$

we obtain the following:

$$\gamma_a\frac{\partial}{\partial t}(D^2 - \alpha^2)w = -\alpha^2 R\theta - (D^2 - \alpha^2)w \tag{14}$$

$$\frac{\partial\theta}{\partial t} = (D^2 - \alpha^2)\theta - (DT_b)w \tag{15}$$

Equations 14 and 15 must now be solved subject to appropriate boundary conditions. Following Caltagirone (1976), we can employ the Galerkin approximation. We write the following:

$$w(z, t) = \sum_{i=1}^{N} a_i(t)w_i(z) \tag{16a}$$

$$\theta(z, t) = \sum_{i=1}^{N} b_i(t)T_i(z) \tag{16b}$$

Substituting into Equations 14 and 15, multiplying the first by $W_j(z)$ and the second by $T_j(z)$, for $j = 1, 2, \ldots, N$ in turn, and integrating with respect to z from 0–1, we get the following:

$$C_{ji}\frac{\mathrm{d}a_i}{\mathrm{d}t} = -D_{ji}a_i + E_{ji}b_i \tag{17a}$$

$$F_{ji}\frac{\mathrm{d}b_i}{\mathrm{d}t} = H_{ji}a_i - G_{ji}b_i \tag{17b}$$

where, with $\langle\cdots\rangle$ written for

$$\int_0^1 \ldots \mathrm{d}z,$$

$C_{ji} = \gamma_a\langle DW_jDW_i + \alpha^2 W_jW_i\rangle$, $D_{ji} = \langle DW_jDW_i + \alpha^2 W_jW_i\rangle$, $E_{ji} = \alpha^2\mathrm{Ra}\langle W_jT_i\rangle$, $F_{ji} = \langle T_jT_i\rangle$, $G_{ji} = \langle DT_jDT_i + \alpha^2 T_jT_i\rangle$, $H_{ji} = \langle(-DT_b)T_jW_i\rangle$.

Writing $x_{2i-1} = a_i$, $x_{2i} = b_i$, $K_{2j-1,2i-1} = C_{ji}$, $K_{2j-1,2i} = 0$, $K_{2j,2i-1} = 0$, $K_{2j,2i} = F_{ji}$, $J_{2j-1,2i-1} = -D_{ji}$, $J_{2j-1,2i} = E_{ji}$, $J_{2j,2i-1} = H_{ji}$, $K_{2j,2i} = -G_{ji}$, we have the matrix equation.

$$\bar{\bar{K}}\frac{\mathrm{d}\bar{x}}{\mathrm{d}t} = \bar{\bar{J}}\bar{x} \tag{18}$$

The interial parameter γ_a is very small in most circumstances, and it is now assumed to be zero.

Stability results

Two "conducting" (isothermal) boundaries

Applying a first-order approximation ($N = 1$) to the case of thermally conducting boundaries, we can take the following as trial functions:

$$W_1 = T_1 = \sin\pi z \tag{19}$$

The calculation of the matrix elements is straightforward but tedious. We find the following:

$$\begin{aligned}\frac{1}{b_1}\frac{\mathrm{d}b_1}{\mathrm{d}t} &= \frac{1}{D_{11}F_{11}}(E_{11}H_{11} - D_{11}G_{11}) \\ &= \frac{\alpha^2}{\pi^2 + \alpha^2}[\mathrm{Ra} - \mathrm{Ra}_I f(\beta)g(t,\tau) - h(\alpha)]\end{aligned} \tag{20}$$

where

$$f(\beta) = \frac{2(1 - e^{-\beta})}{4\pi^2 + \beta^2} \tag{21}$$

$$g(t,\tau) = 1 + \{4\pi^2\varepsilon[4\pi^2\cos\Omega(t-\tau) + \Omega\sin\Omega(t-\tau)]\}/(16\pi^4 + \Omega^2) \tag{22}$$

$$h(\alpha) = (\pi^2 + \alpha^2)^2/\alpha^2 \tag{23}$$

Thus, a disturbance will grow exponentially with time at $t = 0$ if

$$\mathrm{Ra} - \mathrm{Ra}_I f(\beta)g(0,\tau) > h(\alpha) \tag{24}$$

As the wave number α varies, $h(\alpha)$ attains its minimum value $4\pi^2$, when $\alpha = \pi$. This gives the value of the critical wave number for the onset of instability.

The maximum growth rate of a disturbance is attained when $g(0,\tau)$ is at its minimum value. As τ varies, this is attained when

$$\tau = \Omega^{-1}[\pi - \tan^{-1}(\Omega/4\pi^2)] \tag{25}$$

and the minimum value of $g(0,\tau)$ is $1 - 4\pi^2\varepsilon(16\pi^2 + \Omega^2)^{-1/2}$. The critical value of τ corresponds to $t = 0$ being the instant in the heating cycle when the heating is a minimum (i.e., the cooling is a maximum). This is in accord with expectations, because the upper portion of the layer is being cooled (or heated) volumetrically more than the lower portion.

As Ω increases from zero to infinity, the minimum value of $g(0,\tau)$ increases from $1 - \varepsilon$ to 1, and so the stabilizing effect of the volumetric heating increases accordingly. Thus, it is the low-frequency heating that gives rise to the most unstable situation.

It follows that the criterion for instability is that

$$\mathrm{Ra} + \mathrm{Ra}_I f(\beta)[4\pi^2\varepsilon(16\pi^2 + \Omega^2)^{-1/2} - 1] > 4\pi^2 \tag{26}$$

For the case $\mathrm{Ra}_I = 0$, this result reduces to the well-known fact that $\mathrm{Ra}_c = 4\pi^2$ for the HRL problem.

The value of the function $f(\beta)$ increases from zero (when β is zero) to the maximum value 0.402 (when $\beta = 2.36$) and then decreases monotonically to zero (as $\beta \to \infty$). The fact that $f(0) = 0$ means that the effect of uniform volumetric heating does not enter the the stability problem at the first-order approximation; it does enter at the second order. Consequently, at small Ra_I, the effect is proportional to Ra_I^2. This is in accord with the computations of Gasser and Kazimi (1976) for spatially uniform, time-independent volumetric heating.

Bottom boundary "conducting" (isotemperature), top boundary "insulating" (isoflux)

In place of Equation 5, the thermal boundary conditions are now taken to be $T^* = T_0$ at $z^* = 0$, $\partial T^*/\partial z^* = 0$ at $z^* = H$. The dimensionless conduction profile is given by the following:

$$T_b = T_0/T_r + [Qe^{-\beta}/\beta^2]\{\beta e^{\beta} z + 1 - e^{\beta z}\}$$
$$+ [\varepsilon Q e^{i\Omega(t-\tau)}/(\beta^2 - \lambda^2)\cosh\lambda]$$
$$\times \left[\frac{\beta}{\lambda}\sinh\lambda z + e^{-\beta}\cosh\lambda(1-z) - e^{\beta(z-1)}\cosh\lambda\right] \quad (27)$$

Appropriate trial functions are now as follows:

$$W_1 = \sin\pi z, \qquad T_1 = \sin\pi z/2 \quad (28)$$

After performing a considerable amount of algebra, we find that disturbances grow with time provided that

$$\mathrm{Ra}_I\{E(\beta) - F(\beta,\Omega)\cos\Omega(t-\tau) - G(\beta,\Omega)\sin\Omega(t-\tau)\}$$
$$> H(\alpha) \quad (29)$$

where

$$E(\beta) = [8\beta(7\pi^2 + 4\beta^2)/(\pi^2 + 4\beta^2)(9\pi^2 + 4\beta^2)]$$
$$\times \{1 - [3\pi^3 e^{-\beta}/16\beta(7\pi^2 + 4\beta^2)]\} \quad (30)$$

$$F(\beta,\Omega) = (9\pi^3\beta/8)\{[\pi^2/(\pi^2 + 4\beta^2)(\pi^4 + 16\Omega^2)]$$
$$+ [27\pi^2/(9\pi^2 + 4\beta^2)(81\pi^4 + 16\Omega^2)]\}$$
$$+ (9\pi^4 e^{-\beta}/16)\{[\pi^2/(\pi^2 + 4\beta^2)(\pi^4 + 16\Omega^2)]$$
$$- [81\pi^2/(9\pi^2 + 4\beta^2)(81\pi^4 + 16\Omega^2)]\} \quad (31)$$

$$G(\beta,\Omega) = (9\pi^3\beta\Omega/2)\{[1/(\pi^2 + 4\beta^2)(\pi^4 + 16\Omega^2)]$$
$$+ [3/(9\pi^2 + 4\beta^2)(81\pi^4 + 16\Omega^2)]\}$$
$$+ (9\pi^4 e^{-\beta}\Omega/4)\{[1/(\pi^2 + 4\beta^2)(\pi^4 + 16\Omega^2)]$$
$$- [9/(9\pi^2 + 4\beta^2)(81\pi^4 + 16\Omega^2)]\} \quad (32)$$

$$H(\alpha) = 9\pi^2(\pi^2 + \alpha^2)(4\pi^2 + \alpha^2)/256\alpha^2 \quad (33)$$

The critical wave number [that which minimizes $H(\alpha)$] is $2^{-1/2}\pi$.

Bottom boundary "insulating" (isoflux), top boundary "conducting" (isotemperature)

In place of Equation 5, the thermal boundary conditions are now taken to be as follows:

$$\partial T^*/\partial z^* = 0 \quad \text{at} \quad z^* = 0, \qquad T^* = T_0 \quad \text{at} \quad z^* = H \quad (34)$$

The conduction solution, and the stability criterion, can, in each case, be obtained from the previous case (Equations 27–33) by making the transformations $z \to 1 - z$, $\beta \to -\beta$, followed by $Q \to -Q\, e^{-\beta}$, $\mathrm{Ra}_I \to -\mathrm{Ra}_I\, e^{-\beta}$.

Two "insulating" (isoflux) boundaries

In place of Equation 5, the thermal boundary conditions are now as follows:

$$\partial T^*/\partial z^* = 0 \quad \text{at} \quad z^* = 0, H \quad (35)$$

On physical and mathematical grounds, this set of boundary conditions is consistent with a volumetric heat source distribution if, and only if, the time-average of the source term is zero. To have a well-posed problem, we must replace $1 + \varepsilon\, e^{i\omega(t^* - t')}$, the expression within braces in Equation 3, by $\varepsilon\, e^{i\omega(t^* - t')}$. Also, without loss of generality, we can set $\varepsilon = 1$; the effect is the same as incorporating the factor ε into the definition of q'''. The basic temperature distribution is then given (in place of Equation 9) by the following

$$T_b = [Qe^{i\Omega(t-\tau)}/(\beta^2 - \lambda^2)\lambda\sinh\lambda]$$
$$\times \{\beta\cosh\lambda z - \beta e^{-\beta}[\cosh\lambda(1-z) - \lambda e^{\beta(z-1)}\sinh\lambda]\} \quad (36)$$

Suitable trial functions are now $W_1 = \sin\pi z$, $T_1 = 1$. In place of Equation 20–23, we now have the following:

$$\frac{1}{b_1}\frac{db_1}{dt} = -\left[\alpha^2 + \frac{\alpha^2}{\pi^2 + \alpha^2}\mathrm{Ra}_I f(\beta) g(t,\tau)\right] \quad (37)$$

where

$$f(\beta) = 8\beta(1 + e^{-\beta})/(\pi^2 + \beta^2) \quad (38)$$

$$g(t,\tau) = [\pi^2\cos\Omega(t-\tau) + \Omega\sin\Omega(t-\tau)]/(\pi^4 + \Omega^2) \quad (39)$$

The most unstable situation at $t = 0$ is when $\alpha \to 0$, $\tau = \Omega^{-1}\{\pi - \tan^{-1}(\Omega/\pi^2)\}$, and the condition for instability is that

$$\mathrm{Ra}_I > [\pi^2(\pi^2 + \beta^2)(\pi^4 + \Omega^2)^{1/2}]/\beta(1 + e^{-\beta}) \quad (40)$$

(The fact that the critical wavenumber is zero means that the lateral extent of a convection cell is limited only by the presence of lateral walls. This phenomenon, which is peculiar to the choice of isoflux boundary conditions, was discussed in detail in the appendix to Nield (1967).)

As β varies, the minimum value of the right-hand side of this inequality is $7.345(\pi^4 + \Omega^4)^{1/2}$, which is attained when $\beta = 2.63$; as Ω varies the minimum value of this quantity is 72.5, attained when $\Omega \to 0$.

Square-wave time-dependent heating: steady state

So far it has been assumed that the volumetric heat source varies sinusoidally with time. The situation with square-wave (on–off) time dependence is now investigated. The dimensionless temperature for the conduction state is taken to be given by the following:

$$\frac{\partial T_b}{\partial t} = \frac{\partial^2 T_b}{\partial z^2} + Qe^{\beta(z-1)}\{1 + \varepsilon F(t-\tau)\} \quad (41)$$

where

$$F(t-\tau) = \begin{cases} 1 & \text{for} \quad -\pi/\Omega < t - \tau < 0 \\ -1 & \text{for} \quad 0 < t - \tau < \pi/\Omega \end{cases},$$

$$F(t + 2\pi/\Omega) = F(t) \quad (42)$$

A solution is sought subject to the following boundary conditions (appropriate to conducting boundaries at the same temperature)

$$T_b = 0 \quad \text{at} \quad z = 0, 1 \quad (43)$$

Let T_{b-} be the solution in the interval $-\pi/\Omega < t - \tau < 0$, and T_{b+} be that in the interval $0 < t - \tau < \pi/\Omega$. The solutions, each in the form of a steady-state term plus a transient term, are of the following form:

$$T_{b\pm}(z, t-\tau) = \frac{Qe^{-\beta}}{\beta^2}(1 \pm \varepsilon)\{(e^{\beta}-1)z + 1 - e^{\beta z}\} + \sum_{i=1}^{N} b_{n\pm}\exp\{-n^2\pi^2(t-\tau)\}\sin n\pi z. \quad (44)$$

Periodicity requires satisfaction of the following matching conditions:

$$T_{b-}(z,0) = T_{b+}(z,0), \qquad T_{b-}(z, -\pi/\Omega) = T_{b+}(z, \pi/\Omega) \quad (45)$$

These imply the following:

$$\sum_{i=1}^{N}[b_{n-} - b_{n+}]\sin n\pi z = \sum_{i=1}^{N}[b_{n+}E_n - b_{n-}E_n^{-1}]\sin n\pi z = \sum_{i=1}^{N} c_n \sin n\pi z \quad (46)$$

Here $E_n = \exp(n^2\pi^3/\Omega)$, and the last series in Equation 46 is the Fourier expansion of $(2\varepsilon Qe^{-\beta}/\beta^2)[(e^{\beta}-1)z + 1 - e^{\beta z}]$. Thus,

$$b_{n+} = -E_n c_n/(1+E_n), \qquad b_{n-} = c_n/(1+E_n) \quad (47a)$$

where

$$c_n = 4\varepsilon Q[(-1)^{n-1} + e^{-\beta}]/n\pi(\beta^2 + n^2\pi^2) \quad (47b)$$

The conduction-state temperature distribution is given by Equations 44 and 47, and the periodicity requirement. Because $T_{b\pm} = 0$ at $z = 0, 1$, the series in Equation 44 may be differentiated term- by-term.

The stability calculations parallel to those in subsection *Two "conducting" (isothermal) boundaries* may now be carried out. We now have Equation 20, for the case Ra = 0, but with $g(t, \tau)$ replaced by $G(t, \tau)$, where

$$G(t,\tau) = \begin{cases} 1 - \varepsilon + \dfrac{2\varepsilon\exp[-4\pi^2(t-\tau)]}{1+\exp(4\pi^3/\Omega)} & \text{for } -\pi/\Omega < t-\tau < 0 \\ 1 + \varepsilon - \dfrac{2\varepsilon\exp[-4\pi^2(t-\tau)]}{1+\exp(-4\pi^3/\Omega)} & \text{for } 0 < t-\tau < \pi/\Omega \end{cases} \quad (48)$$

The largest growth rate for disturbances occurs at $t = 0$, when $G(0, \tau)$ takes its minimum value. This occurs when $\tau = 0$, and the minimum value is as follows:

$$G(0,0) = 1 - \varepsilon\tanh(2\pi^3/\Omega) \quad (49)$$

Thus, the most unstable situation occurs at the end of the cooling portion of the cycle, as expected. It is noteworthy that $G(0,0)$ is positive, and so the conduction state is stable for all values of Ra_I, if $\Omega > 2\pi^3/\tanh^{-1}(\varepsilon^{-1})$. As Ω varies, the most unstable disturbances are those for which the frequency Ω is small, and then for instability we must have $\varepsilon > 1$. The criterion for instability is that:

$$\mathrm{Ra}_I > h(\alpha)/f(\beta)[\varepsilon\tanh(2\pi^3/\Omega) - 1] \quad (50)$$

Because the smallest value of $h(\alpha)$ is $4\pi^2$, the critical internal Rayleigh number is given by the following:

$$\mathrm{Ra}_{Ic} = 4\pi^2/f(\beta)[\varepsilon\tanh(2\pi^3/\Omega) - 1] \quad (51)$$

Comparison with the corresponding result for the sinusoidal case (Equation 26 with Ra = 0) shows that the square-wave case leads to a more unstable situation that the corresponding sinusoidal case with the same amplitude, as we might expect. The difference tends to zero as Ω tends to zero, as expected.

Square-wave time-dependent heating: transient problem

The following transient situation is now considered. Suppose that when $t - \tau = -\pi/\Omega$ the temperature is zero throughout the porous medium, and at this instant the square-wave heating cycle of the previous section is begun. The form of the conduction-state solution is still given by Equation 45. We now need the following Fourier expansion:

$$\frac{Qe^{-\beta}}{\beta^2}(1 \pm \varepsilon)\{(e^{\beta}-1)z + 1 - e^{\beta z}\} = \sum_{i=1}^{N} d_{n\pm}\sin n\pi z \quad (52)$$

so that

$$T_{b\pm}(z, t-\tau) = \sum_{i=1}^{N}\{d_{n\pm} + b_{n\pm}\exp[-n^2\pi^2(t-\tau)]\}\sin n\pi z \quad (53)$$

Here $d_{n\pm} = (1 \pm \varepsilon)c_n/2\varepsilon$, where c_n is given by Equation 47b.

Because $T_{b-}(z, -\pi/\Omega) = 0$ for all values of z, we must have $b_{n-} = -d_n/E_n$. The most unstable situation is, again, at the end of the cooling phase; namely when $t - \tau = 0$. The conduction-state temperature distribution is then given by

$$T_b = \sum_{i=1}^{N} d_{n-}[1 - E_n^{-1}]\sin n\pi z$$

The critical internal Rayleigh number is now given by

$$\mathrm{Ra}_{Ic} = 4\pi^2/f(\beta)(\varepsilon - 1)[1 - \exp(-4\pi^3/\Omega)] \quad (54)$$

For all values of ε and Ω, this critical number is less than that given by Equation 51. Thus, the transient case gives rise to a more unstable situation than does the steady-state square-wave case. This is as expected, because in the steady-state case, the effect of cooling is partially offset by the heating earlier in the cycle.

Discussion and conclusions

In this paper, which is a pioneering investigation of the onset of convection in a porous medium induced by nonuniform volumetric heating, the author has been content to apply a first-order Galerkin approximation, in order to rapidly investigate a large parameter space (α, β, ε, τ, Ω, Ra, Ra_I), for various combinations of thermal boundary conditions and various types of waveform for the time-periodicity. At present, there are no physical experimental results known to the author for which a precise check of the theory is possible. When some are available, more accurate calculations will be desirable. In the meantime, the present results provide a good qualitative picture of the stability–instability boundary. The criteria in this paper provide sufficient conditions for instability (i.e., upper bounds on the appropriate critical Rayleigh numbers), because they specify conditions that at least one disturbance (that described by the trial function chosen) grows exponentially. In other words, the stability domain cannot be larger than that delineated here.

Besides the stability criteria for the individual cases given in the second through the sixth sections, general results have been obtained. It has been demonstrated that the most unstable conduction-state temperature profile occurs at the end of the cooling phase of a period cycle if the decay parameter β is positive, that the square-wave time-periodic source leads to a more unstable situation than a sinusoidal time-periodic source of the same amplitude, and that the transient on–off heating case leads to greater instability than the corresponding steady state.

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